Uncertainty Quantification for the Relativistic Inverse Stellar Structure Problem

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Overview

Neutron-Star Equation of State

- Neutron stars are the most compact objects in the universe;
- Collapsed core of a massive supergiant star.
- Equation of state (EoS) relates P and ϵ;
- EoS of neutron stars is poorly understood; Exceed lab limits;

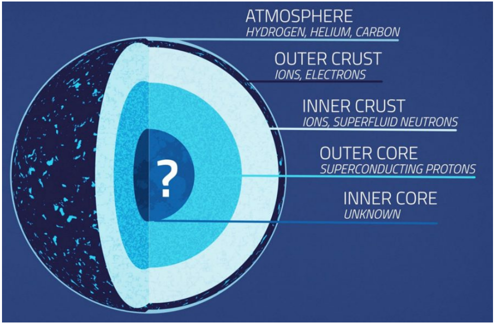


Figure: Neutron-star inner structure. Credit to NASA.

How to Improve Our Understanding of NS EoS?

- Macroscopic observables (e.g. M and R) are critical;
- Neutron Star Interior Composition Explorer (NICER) is an X-ray telescope on the International Space Station;
- Observe the soft thermal X-rays emitted by neutron stars;
- Constrain M and R to $10 - 20\%$ uncertainty.

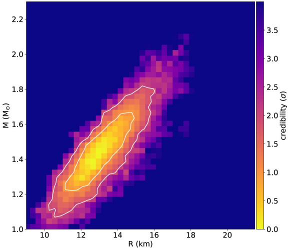


Figure: PSR J0030+0451 M and R constraints from NICER (Miller et al., 2019).

Relativistic Stellar Structure Problem

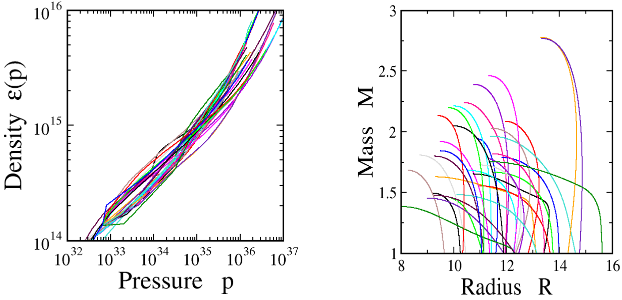
- Oppenheimer and Volkoff provided a solution of the Einstein field equation for a spherically symmetric star in static equilibrium (Oppenheimer and Volkoff, 1939):

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \tag{1.1}$$

$$\frac{dP}{dr} = -(\epsilon + P)\frac{m + 4\pi r^3 P}{r(r-2m)}; \tag{1.2}$$

- Input: Central pressure (P_c) and EoS;
- Output: M and R for a neutron star with specific P_c;
- Pick a specific model of EoS, vary P_c values, and get a set of $[M_i, R_i]$ values on a curve.

Relativistic Stellar Structure Problem



(a) EoS models. (b) Mass-Radii pairs.

Figure: One-to-one relationship between neutron-star EoS and M-R curve.

Relativistic Inverse Stellar Structure Problem

- There is a one-to-one mapping relationship between EoS and M-R curves;
- Deduce EoS from macroscopic observables like M and R;
- Spectral method is proved to be efficient for representing EoS (Lindblom and Indik, 2012);
- The noise in the observational data is interesting;
- Add noise for 0.1%, 1%, 10% and 20% to the data.

Causal Pressure-Based Spectral Representation

- In a barotropic fluid, the speed of sound v is related to EoS by: $v^2 = dp/d\epsilon$ (Lindblom, 2022);
- Define a velocity function Υ, which preserves the causality:

$$\Upsilon = \frac{c^2 - v^2}{v^2}; \tag{2.1}$$

- The EoS becomes:

$$\epsilon(P) = \epsilon_0 + (P - P_0) + \int_{P_0}^{P} \Upsilon(P')\, dP'; \tag{2.2}$$

- Get a parametric representation by expressing $\Upsilon(P, \upsilon_k)$ in spectral form:

$$\Upsilon(P, \upsilon_k) = \Upsilon_0 \exp\left\{ \sum_{k=0}^{N_{\text{parms}}-1} \upsilon_k (1+z)\, T_k(z) \right\}. \tag{2.3}$$

Fitting the Spectral Parameters

- We have $[M_i, R_i]$ from observation;
- We could compute $[M(P_c^i, \upsilon_k), R(P_c^i, \upsilon_k)]$ from Oppenheimer-Volkoff Equations;
- We could find best-fit of computed data to observed data by minimizing the modeling errors for mass and radius (Lindblom and Indik, 2012);
- Find the best fit central pressure and spectral parameters.

Noisy Mass and Radius Data

- Use GM1L equation of state to generate fake mass and radius data (Typel et al., 2010);
- Use a random number generator to add 0.1%, 1%, 10% and 20% noise to computed datasets.

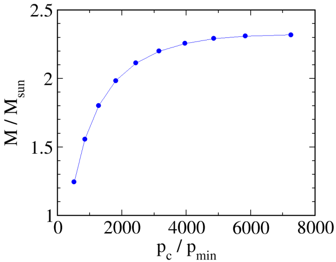


Figure: 10 central pressures are selected to set masses ranging from $1.2M_\odot$ to $M_{\max}$.

Brief Overview

- Relativistic inverse stellar structure problem;
- Noisy data for mass and radius $\rightarrow$ parametric representation equation of state;
- Best fitted central pressure and spectral parameters;
- Comparing modelling error χ (between mass and radius) and Δ (between exact EoS and spectral EoS).

Modeling Error χ for Mass and Radius Data

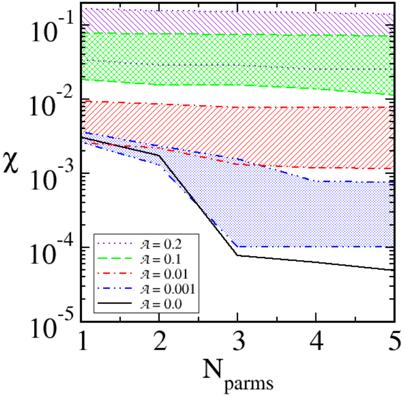


Figure: Modeling error χ for mass and radius data.

Modeling Error Δ for Equation of State

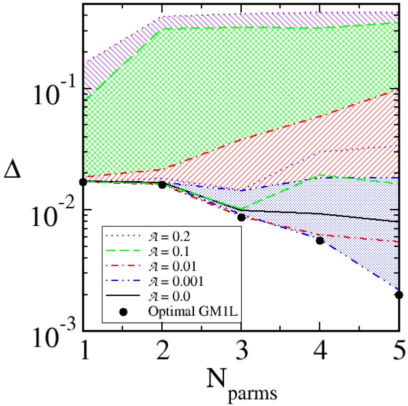


Figure: The range of modeling error Δ for the equation of state.

Qualitative Argument

- The upper limit for Δ value is meaningful:
- Constrain EoS to 2% by using 1 or 2 spectral parameters with 1% noise in data,
- Or constrain EoS to 1% by using 3 spectral parameters with 0.1% noise in data.
- Going to more parameters does not always improve the result:
- $N_{\text{parms}} = 1$ is sufficient for 10% and 20% noise;
- $N_{\text{parms}} = 3$ is optimal for 0.1% noise.

Conclusion and Next Steps

- Currently, most observations are around $10 - 20\%$ level of uncertainty (Chatziioannou et al., 2024) ;
- Understand the ability and limit for spectral method in relativistic inverse stellar structure problem;
- Future directions: modifying selections of P_c,
- Changing different EoS models,
- And increasing the number of $[M_i, R_i]$ pairs.

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Any Questions?

Difference between two modeling errors?

Explain more about EoS modeling error figure?

Any other ways to constrain EoS?